

University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO PROBLEM SET 8

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1. K&K problem 8.5

A gyroscope with mass M has angular velocity ω_s and moment of inertia I_s . It pivots at one end, and the center of mass is a distance l from the pivot. The angular momentum of the gyroscope is thus

$$L = I_s \omega_s$$

The gyroscope undergoes an acceleration a perpendicular to the spin axis. The fictitious force will create a torque of magnitude

$$\tau = Mal$$

The direction of this torque is perpendicular to both the acceleration and the gyroscope axis (down in the figure), causing the gyroscope axis to precess in the direction indicated by the angle θ . The magnitude of the angular momentum will not change, but the direction will. Thus the gyroscope axis will rotate around the direction of acceleration. The rate at which this happens is ω , and

$$\frac{dL}{dt} = L\omega = \tau$$

This gives the following relation

$$Mla(t) = I_s \omega_s \omega(t)$$

Both the acceleration and the angular velocity can depend on time. If we integrate both sides of this equation, we can get a relation between the final velocity and the total angle of rotation. The integral of the acceleration is just the velocity and the integral of the angular velocity is just the angle:

$$Mlv = I_s \omega_s \theta \Rightarrow v = \frac{I_s \omega_s}{Ml} \theta$$

2. K&K problem 8.11

A hydrofoil moves with respect to the earth's surface at the equator with velocity $\mathbf{v} = 200$ mi/hr directed along each of the four points of the compass. At rest with respect to the surface of the earth, the acceleration of gravity is \mathbf{g} . We are asked to find the effective gravitational acceleration \mathbf{g}' that is felt by a passenger (of mass m) who is at rest with respect to the hydrofoil.

The (fictitious) Coriolis force on the passenger, which is proportional to the passenger's (vanishing) velocity in the hydrofoil's frame, must be zero if evaluated in this frame. The (fictitious) centrifugal force on the passenger is

$$\begin{aligned} \mathbf{F}_{\text{cent}} &= -m\boldsymbol{\Omega}' \times (\boldsymbol{\Omega}' \times \mathbf{R}) \\ &= -m(\boldsymbol{\Omega}'(\boldsymbol{\Omega}' \cdot \mathbf{R}) - \mathbf{R}(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}')) \\ &= m(-\boldsymbol{\Omega}'(\boldsymbol{\Omega}' \cdot \mathbf{R}) + \mathbf{R}\boldsymbol{\Omega}'^2) \end{aligned}$$

where $\boldsymbol{\Omega}'(\mathbf{v})$ is the *total* angular velocity of the passenger, due *both* to the rotation of the earth and to the motion of the hydrofoil; $\mathbf{R} = R\hat{\mathbf{x}}$ is a vector pointing from the earth's center to the hydrofoil at the equator; and the "bac cab" rule is applied to the first line. The hydrofoil's velocity has one of the four directions (E,W,N,S) = $(\hat{\mathbf{y}}, -\hat{\mathbf{y}}, \hat{\mathbf{z}}, -\hat{\mathbf{z}})$, yielding an angular velocity $\boldsymbol{\omega}$ due to hydrofoil motion relative to the earth's surface:

$$\boldsymbol{\omega} = \frac{v}{R}(\hat{\mathbf{z}}, -\hat{\mathbf{z}}, -\hat{\mathbf{y}}, \hat{\mathbf{y}})$$

To this one must add the earth's angular velocity

$$\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$$

in order to get the total angular velocity $\boldsymbol{\Omega}' = \boldsymbol{\omega} + \boldsymbol{\Omega}$ of the hydrofoil. Evidently $\boldsymbol{\Omega}'$ is perpendicular to \mathbf{R} , so

$$\mathbf{F}_{\text{cent}}(\mathbf{v}) = \hat{\mathbf{x}}mR(\boldsymbol{\Omega}'(\mathbf{v}))^2$$

Therefore \mathbf{g}' points along $-\hat{\mathbf{x}}$, *i.e.* toward the earth's center, for all four directions (E,W,N,S) of \mathbf{v} , as does \mathbf{g} . Thus

$$\frac{g' - g}{g} \equiv \frac{\Delta g}{g} = \frac{-R((\boldsymbol{\Omega}')^2 - \Omega^2)}{g}$$

For these four directions,

$$\begin{aligned}\Omega'^2 - \Omega^2 &= 2\Omega\omega + \omega^2 \quad (\text{E}) \\ &= -2\Omega\omega + \omega^2 \quad (\text{W}) \\ &= \omega^2 \quad (\text{N and S})\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\Delta g}{g} &= \frac{R}{g}(-2\Omega\omega - \omega^2) \quad (\text{E}) \\ &= \frac{R}{g}(2\Omega\omega - \omega^2) \quad (\text{W}) \\ &= \frac{R}{g}(-\omega^2) \quad (\text{N and S})\end{aligned}$$

Using $\omega/\Omega = 0.1931$ and $R\Omega^2/g = 0.003432$, we calculate $|\Delta g/g| = 0.001325$ from the $2\Omega\omega$ term and $|\Delta g/g| = 0.000128$ from the ω^2 term. Thus

$$\begin{aligned}\Delta g/g &= -0.001453 \quad (\text{E}) \\ &= +0.001197 \quad (\text{W}) \\ &= -0.000128 \quad (\text{N and S})\end{aligned}$$

3. K&K problem 9.3

A particle moves in a circle under the influence of an inverse cube law force. This means that the potential is inverse squared and it is attractive. The effective potential is given by

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{A}{r^2}$$

The radial force is zero for a circular orbit, so we can find the radius.

$$0 = \frac{dU_{\text{eff}}}{dr} = -\frac{L^2}{mr^3} + \frac{2A}{r^3}$$

This shows that a circular orbit can have any radius, but there is only one possible magnitude of angular momentum, given by

$$L^2 = 2Am$$

Plugging this value of the angular momentum into the effective potential, we find the peculiar result that

$$U_{\text{eff}} = 0$$

Since U_{eff} is constant, $d^2r/dt^2 = 0$, so if the particle acquires a nonzero radial velocity it will continue with the same radial velocity. If the particle moves with uniform radial velocity v_r , the following equations are satisfied

$$\frac{dr}{dt} = v_r \quad \frac{d\theta}{dt} = \frac{L}{mr^2(t)}$$

Solving the first is easy: $r(t) = r_0 + v_r t$. Plugging this result into the second equation, we find

$$\frac{d\theta}{dt} = \frac{L}{m(r_0 + v_r t)^2}$$

We can solve this equation by direct integration, assuming that $\theta(0) = 0$:

$$\theta(t) = \int_0^t \frac{L dt}{m(r_0 + v_r t)^2} = \frac{L}{mv_r} \left(\frac{1}{r_0} - \frac{1}{r(t)} \right)$$

We replace L with $\sqrt{2mA}$ to get the final answer

$$\theta(t) = \frac{1}{v_r} \sqrt{\frac{2A}{m}} \left(\frac{1}{r_0} - \frac{1}{r(t)} \right)$$

4. K&K problem 9.4

A particle moves in a circular orbit in the potential $U = -A/r^n$. We want to know for which values of n the orbit is stable. The effective potential is given by

$$U_{\text{eff}} = \frac{L^2}{2mr^2} - \frac{A}{r^n}$$

To find the circular orbit radius we evaluate $dU_{\text{eff}}/dr = 0$:

$$\frac{dU_{\text{eff}}}{dr} = -\frac{L^2}{mr^3} + \frac{nA}{r^{n+1}}$$

This gives the radius of the circular orbit r_0 when we set it to zero.

$$r_0^{n-2} = \frac{nAm}{L^2}$$

Since r_0^{n-2} must be a positive quantity for any value of n , and $A > 0$, this equation requires

$$n > 0$$

We now look at the second derivative of the effective potential at $r = r_0$. If it is positive, then it is a potential minimum and the orbit is stable.

$$\begin{aligned}\frac{d^2 U_{\text{eff}}}{dr^2} &= \frac{3L^2}{mr^4} - \frac{n(n+1)A}{r^{n+2}} \\ &= \frac{1}{r^4} \left(\frac{3L^2}{m} - \frac{n(n+1)A}{r^{n-2}} \right) > 0\end{aligned}$$

Since r is always greater than zero we can divide it away. Substituting for r^{n-2} at $r = r_0$,

$$\frac{3L^2}{m} - \frac{(n+1)L^2}{m} > 0 \Rightarrow n < 2$$

Putting both inequalities together,

$$0 < n < 2$$

Recall from the previous problem that when $n = 2$ the motion is barely unstable. When $n = 0$, U is constant, so there is no attractive force, therefore no circular orbit: this case is also unstable.

5. K&K problem 9.6

A particle moves in an attractive central force Kr^4 with angular momentum l . If it moves in a circular orbit with radius r_0 , the central force must provide exactly the necessary centripetal acceleration:

$$\frac{mv^2}{r_0} = Kr_0^4 = \frac{l^2}{mr_0^3} \Rightarrow r_0^7 = \frac{l^2}{mK}$$

Relative to $r = 0$, the energy of the orbit is

$$E = \frac{1}{2}mv^2 + \frac{1}{5}Kr_0^5$$

where we have integrated the force to get the potential. Plugging in $v^2 = Kr_0^5/m$, we get

$$E = \frac{1}{2}Kr_0^5 + \frac{1}{5}Kr_0^5 = \frac{7}{10}Kr_0^5$$

Substituting the above value for r_0 , we get the final result for the energy (relative to $r = 0$):

$$E = \frac{7}{10}K \left(\frac{l^2}{mK} \right)^{5/7}$$

To find the frequency of small radial oscillations, we must evaluate the second derivative of the effective potential. Remember that for small oscillations the effective spring constant k for radial motion is

$$k = \left. \frac{d^2 U_{\text{eff}}}{dr^2} \right|_{r_0}$$

The effective potential is

$$U_{\text{eff}} = \frac{l^2}{2mr^2} + \frac{1}{5}Kr^5$$

The second derivative is easily found

$$\frac{d^2 U}{dr^2} = \frac{3l^2}{mr^4} + 4Kr^3$$

Plugging in $l^2 = m^2 v^2 r^2 = mKr_0^7$, we get

$$\left. \frac{d^2 U}{dr^2} \right|_{r_0} = 7Kr_0^3$$

Substituting the value of r_0 , we find the effective spring constant k and the angular frequency ω of radial oscillation about the stable circular orbit:

$$\begin{aligned}k &= m\omega^2 = 7K \left(\frac{l^2}{mK} \right)^{3/7} \\ \omega &= \sqrt{\frac{7K}{m}} \left(\frac{l^2}{mK} \right)^{3/14}\end{aligned}$$

6. K&K problem 9.12

A spacecraft of mass m orbits the earth at a radius $r = 2R_e$. It will transfer to another circular orbit with radius $r = 4R_e$.

(a.) We know the radius of each orbit, so we can easily find the energies of the two orbits. The energy of a bound orbit in a $1/r$ potential is given by

$$E = -\frac{GMm}{A}$$

where A is the major axis of the elliptical (here the diameter of the circular) orbit. We can use this to find the energies of the two orbits. The values of A are simply $4R_e$ for the first and $8R_e$

for the second. The energy input needed to go from one orbit to the other is at least

$$\Delta E = -\frac{GMm}{R_e} \left(\frac{1}{8} - \frac{1}{4} \right) = \frac{GMm}{8R_e}$$

Plugging in the values given,

$$\Delta E = 2.34 \times 10^{10} \text{ joules}$$

(b.) At point A , the rocket is fired, putting the spacecraft in an elliptical orbit. The major axis of this orbit is $A = 6R_e$. To find the initial speed, we use the energy equation. The energy is partly gravitational potential energy and partly kinetic energy:

$$E = \frac{1}{2}mv_0^2 - \frac{GMm}{2R_e} = -\frac{GMm}{4R_e}$$

Solving this equation for v_0 , we find the orbital speed

$$v_0 = \sqrt{\frac{GM}{2R_e}}$$

The energy of the elliptical orbit is given by

$$E = -\frac{GMm}{6R_e} \Rightarrow \Delta E = \frac{GMm}{12R_e}$$

The change in energy at point A is entirely due to a change in speed.

$$\Delta E = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 \Rightarrow v_1 = \sqrt{\frac{2GM}{3R_e}}$$

The change in speed required at point A is thus

$$\Delta v_A = \left(\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}} \right) \sqrt{\frac{GM}{R_e}} = 865 \text{ m/sec}$$

We repeat this analysis at point B . Conservation of angular momentum gives

$$2R_e v_1 = 4R_e v_2 \Rightarrow v_2 = \sqrt{\frac{GM}{6R_e}}$$

The energy of the new circular orbit is given by

$$E = -\frac{GMm}{8R_e} \Rightarrow \Delta E = \frac{GMm}{24R_e}$$

Again this is due to the change in speed.

$$\Delta E = \frac{1}{2}mv_3^2 - \frac{1}{2}mv_2^2 \Rightarrow v_3 = \sqrt{\frac{GM}{4R_e}}$$

Finally we obtain the change in speed at point B

$$\Delta v_B = \left(\frac{1}{2} - \sqrt{\frac{1}{6}} \right) \sqrt{\frac{GM}{R_e}} = 726 \text{ m/sec}$$

Since the two velocities at point A are both tangent to each other, and similarly for point B , the only changes in the velocities at either point are the changes in their magnitudes.

7. A satellite of mass m moves in a circular orbit of radius R at speed v . It is influenced by the gravity of a fixed mass at the origin.

(a.) The mechanical energy of the satellite is given by

$$E = \frac{1}{2}mV^2 - \frac{GMm}{R}$$

We know that gravity exactly provides the centripetal acceleration.

$$\frac{GMm}{R^2} = \frac{mV^2}{R} \Rightarrow \frac{GM}{R} = V^2$$

The total energy is thus

$$E = \frac{1}{2}mV^2 - mV^2 = -\frac{1}{2}mV^2$$

(b.) At a certain point on the orbit, the direction of travel of the satellite changes. The magnitude of the velocity does not change, so the total energy of the orbit doesn't change. We can now find the kinetic energy at closest approach:

$$\begin{aligned} E &= -\frac{1}{2}mV^2 = \frac{1}{2}mv^2 - \frac{5GMm}{R} \\ &= \frac{1}{2}mv^2 - 5mV^2 \\ &\Rightarrow v = 3V \end{aligned}$$

(c.) The circular orbit has angular momentum $L_1 = mRV$, while the elliptical orbit's angular momentum, evaluated at the perigee, is

$$L_2 = m\frac{R}{5}3V = \frac{3}{5}L_1$$

Therefore, just after the transition from circular to elliptical orbit, a fraction $\frac{3}{5}$ of the original velocity must remain tangential, while $\frac{4}{5}$ of it becomes radial (the squares of the two fractions must add to unity according to Pythagoras). Therefore the satellite turns through an angle

$$\alpha = \arctan \frac{4/5}{3/5} = 53.1^\circ$$

8. A spaceship is moving on a circular path that will take it directly through a gas cloud. The angular momentum with respect to the gas cloud is measured to be constant. We want to know what attractive central force causes this. Immediately we notice that as the ship passes through the center of the cloud, its velocity must become infinite, because the angular momentum $l = mvr$ is conserved. If l were zero, the ship could only fall straight into the cloud.

We can express the circular trajectory of the ship as a function of θ by inspection:

$$r(\theta) = 2R \cos \theta \quad (-\pi/2 < \theta < \pi/2)$$

Take ϕ to be the azimuth of the spaceship on the circle ($-\pi < \phi < \pi$), with $\phi \equiv 0$ when $\theta = 0$. Consider the isosceles triangle with sides r , r , and R . Requiring its angles to add up to π , it is easy to see that $\phi = 2\theta$.

We are given two definite facts. One is that the ship's angular momentum about the center of the cloud

$$L = mr^2\dot{\theta} = 4mR^2\dot{\theta}\cos^2\theta$$

is constant. (This expression confirms our previous observation that the ship's velocity $2R\dot{\theta}$ must be infinite at the center of the cloud, where $\cos\theta = \cos\pi/2 = 0$.) The second fact is that the spaceship moves in a circle of radius R . The centripetal force $mR\dot{\phi}^2$ required to keep it in circular motion must be supplied by the component along \mathbf{R} of the unknown attractive force F :

$$F \cos \theta = mR\dot{\phi}^2 = 4mR\dot{\theta}^2$$

Using the previous equation for L to eliminate $\dot{\theta}$ from this equation,

$$F \cos \theta = 4mR \frac{L^2}{16m^2R^4 \cos^4 \theta}$$

$$F = \frac{L^2}{4mR^3 \cos^5 \theta}$$

Finally, using the first equation to eliminate $\cos \theta$,

$$F = \frac{L^2 32 R^5}{4mR^3 r^5}$$

$$F = \frac{8L^2 R^2}{mr^5}$$

Since L and R are constant, the unknown attractive force depends on the inverse fifth power of the spaceship's separation from the center of the cloud, for this particular spaceship trajectory. The last equation is the desired result. However, this is no simple force field: its coupling to the spaceship is contrived to depend quadratically both upon the spaceship's angular momentum about the cloud's center and upon the radius of its circular orbit.

As an alternative to considering the centripetal force that must be supplied by F , one can hypothesize that F is a conservative as well as a central force. (At least it is clear from the fact that the spaceship's orbit is closed that there can be no *monotonic* decrease or increase in the total energy E). From the above equation for L , one readily sees that the ship's speed $v = R\dot{\phi} = 2R\dot{\theta}$ is proportional to r^{-2} . Therefore the ship's kinetic energy K is proportional to r^{-4} . If E is to be conserved, the potential energy U also must be proportional to r^{-4} so that it can cancel the r dependence of K ; its radial derivative $-dU/dr = F_r$ must then be proportional to r^{-5} . The constant of proportionality is easily verified to be the same as is given above.